EXCEPTIONAL REGIONS AND ASSOCIATED EXCEPTIONAL HYPERBOLIC 3-MANIFOLDS

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Abstract. A closed hyperbolic 3-manifold is exceptional if its shortest geodesic does not have an embedded tube of radius $\ln(3)/2$. D. Gabai, R. Meyerhoff and N. Thurston identified seven families of exceptional manifolds in their proof of the homotopy rigidity theorem. They identified the hyperbolic manifold known as Vol3 in the literature as the exceptional manifold associated to one of the families. It is conjectured that there are exactly 6 exceptional manifolds. We find hyperbolic 3-manifolds, some from the SnapPea's census of closed hyperbolic 3-manifolds, associated to 5 other families. Along with the hyperbolic 3-manifold found by Lipyanskiy associated to the seventh family we show that any exceptional manifold is covered by one of these manifolds. We also find their group coefficient fields and invariant trace fields.

1. Introduction

A closed hyperbolic 3-manifold is *exceptional* if its shortest geodesic does not have an embedded tube of radius $\ln(3)/2$. Exceptional manifolds arise in the proof of the rigidity theorem proved by D. Gabai, R. Meyerhoff and N. Thurston in [6]. It is conjectured that there are exactly 6 exceptional manifolds.

Let N be a closed hyperbolic 3-manifold and δ be the shortest geodesic in N. If δ does not have an embedded tube of radius $\ln(3)/2$ then there is a two-generator subgroup G of $\pi_1(N)$ such that $\mathbb{H}^3/G$ also has this property. Assume G is generated by f and w, where $f \in \pi_1(N)$ is a primitive hyperbolic isometry whose fixed axis $\delta_0 \in \mathbb{H}^3$ projects to δ and $w \in \pi_1(N)$ is a hyperbolic isometry which takes δ_0 to its nearest translate. Thus, it is necessary to study two-generator subgroups Γ of $\mathrm{PSL}(2,\mathbb{C})$ with the property that one of the generators is the shortest geodesic in $\mathbb{H}^3/\Gamma$ and the distance from its nearest translate is less than $\ln(3)$.

The space of two-generator subgroups of $\mathrm{PSL}(2,\mathbb{C})$ is analysed in the proof of the rigidity theorem of [6]. The rigidity theorem is proved using Gabai's

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theorem [5], which states that the rigidity theorem is true if some closed geodesic has an embedded tube of radius $\ln(3)/2$. The authors of [6] show that this holds for all but seven exceptional families of closed hyperbolic 3-manifolds. These seven families are handled separately. The seven families are obtained by parametrizing the space of two-generator subgroups of $\mathrm{PSL}(2,\mathbb{C})$ by a subset of $\mathbb{C}^3$, dividing the parameter space into about a billion regions and eliminating all but seven regions. These seven regions correspond to the seven exceptional families and are known as *exceptional regions.* They are denoted by X_i for $i = 0, \ldots, 6$ and described as boxes in $\mathbb{C}^3$. For example, for the region X_3 see Table 1.

Parameter	Range $Re(Parameter)$	Range $Im(Parameter)$
L'	0.58117 to 0.58160	-3.31221 to -3.31190
D'	1.15644 to 1.15683	-2.75628 to -2.75573
R'	1.40420 to 1.40454	-1.17968 to -1.17919

TABLE 1. Parameter ranges for the region X_3

A *quasi-relator* in a region is a word in f, w, f^{-1} and w^{-1} that is close to the identity throughout the region and experimentally appears to converge to the identity at some point. Table 2 gives the two quasi-relators specified for each region X_i in [6]. The group $G_i = \langle f,\ w | r_1(X_i),\ r_2(X_i)\rangle$, where $r_1(X_i),\ r_2(X_i)$ are the quasi-relators for X_i, is called the *marked group* for the region X_i. It follows from [6] that any exceptional manifold has a two-generator subgroup of its fundamental group whose parameter lies in one of the exceptional regions.

Let $\rho(x, y)$ denote the hyperbolic distance between $x,\ y \in \mathbb{H}^3$. For an isometry f of $\mathbb{H}^3$ define $Relength(f) = \inf\{\rho(x, f(x)) |\ x \in \mathbb{H}^3\}$. Let T consists of those parameters corresponding to the groups $\{G,\ f,\ w\}$ such that $Relength(f)$ is the shortest element of G and the distance between the axis of f and its nearest translate is less than $\ln(3)$. Let $S = exp(T)$. In [6] the authors made the following conjecture.

Conjecture 1. *Each sub-box X_i, $0 \le i \le 6$ contains a unique element s_i of S. Further if $\{G_i, f_i, w_i\}$ is the marked group associated to s_i then $N_i = \mathbb{H}^3/G_i$ is a closed hyperbolic 3-manifold with the following properties*

(i) *N_i has fundamental group $\langle f,\ w | r_1(X_i),\ r_2(X_i)\rangle$ where r_1 and r_2 are the quasi-relators associated to the box X_i.*
(ii) *N_i has a Heegaard genus 2 splitting realizing the above group presentation.*
(iii) *N_i nontrivially covers no manifold.*
(iv) *N_6 is isometric to N_5.*
(v) *If (L_i, D_i, R_i) is the parameter in T corresponding to s_i, then L_i, D_i, R_i are related as follows:*

Region	Quasi-Relators
X_0	$r_1 = fwf^{-1}w^2f^{-1}wfw^2$ $r_2 = f^{-1}wfwfw^{-1}fwfw$
X_1	$r_1 = f^{-2}wf^{-1}w^{-1}f^{-1}w^{-1}fw^{-1}f^{-1}w^{-1}f^{-1}wf^{-2}w^2$ $r_2 = f^{-2}w^2f^{-1}wfwfw^{-1}fwfwf^{-1}w^2$
X_2	$r_1 = f^{-1}wfwfw^{-1}f^2w^{-1}fwfwf^{-1}w^2$ $r_2 = f^{-2}wf^{-2}w^2f^{-1}wfwfwf^{-1}w^2$
X_3	$r_1 = f^{-2}wfwf^{-2}w^2f^{-1}w^{-1}f^{-1}wf^{-1}w^{-1}(fw^{-1}f^{-1}w^{-1}f)^2$ $w^{-1}f^{-1}wf^{-1}w^{-1}f^{-1}w^2$ $r_2 = f^{-2}wfwf^{-1}wf(w^{-1}fwfw^{-1})^2fwf^{-1}wfwf^{-2}w^2$ $f^{-1}w^{-1}f^{-1}w^2$
X_4	$r_1 = f^{-2}wfwf^{-1}(wfw^{-1}f)^2wf^{-1}wfwf^{-2}w^2$ $(f^{-1}w^{-1}f^{-1}w)^2w$ $r_2 = f^{-1}(f^{-1}wfw)^2f^{-2}w^2f^{-1}w^{-1}f^{-1}w(f^{-1}w^{-1}fw^{-1})^2$ $f^{-1}wf^{-1}w^{-1}f^{-1}w^2$
X_5	$r_1 = f^{-1}wf^{-1}w^{-1}f^{-1}wf^{-1}wfwfw^{-1}fwfw$ $r_2 = f^{-1}wfwfw^{-1}fw^{-1}f^{-1}w^{-1}fw^{-1}fwfw$
X_6	$r_1 = f^{-1}w^{-1}f^{-1}wf^{-1}w^{-1}f^{-1}w^{-1}fw^{-1}fwfw^{-1}fw^{-1}$ $r_2 = f^{-1}w^{-1}fw^{-1}fwfwf^{-1}wfwfw^{-1}fw^{-1}$

TABLE 2. Quasi-relators for all the regions

For $X_0,\ X_5,\ X_6$: $L = D,\ R = 0$.
For $X_1,\ X_2,\ X_3,\ X_4$: $R = L/2$.

D. Gabai, R. Meyerhoff and N. Thurston [6] proved that Vol3, the closed hyperbolic 3-manifold with conjecturally the third smallest volume, is the unique exceptional manifold associated to the region X_0. Jones and Reid [12] proved that Vol3 does not nontrivially cover any manifold and that the exceptional manifolds associated to the regions X_5 and X_6 are isometric.

In this paper we investigate the seven exceptional regions and the associated exceptional hyperbolic 3-manifolds. In Section 3, using Newton's method for finding roots of polynomials in several variables, we solve the equations obtained from the entries of the quasi-relators to very high precision. Then, using the program PARI-GP, [8] we find entries of the generating matrices as algebraic numbers, find the group coefficient fields and verify with exact arithmetic that the quasi-relators are relations for all the regions. We also find the invariant trace fields for all the groups verifying and extending the data given in [12]. In Section 4 we show that the manifolds $v2678(2,1)$, $s778(-3,1)$ and $s479(-3,1)$ from SnapPea's census of closed hyperbolic 3-manifolds [18] are the exceptional manifolds associated to the regions $X_1,\ X_2,\ X_5$ and X_6 respectively. We also show that their fundamental groups are isomorphic to the marked groups G_i for $i = 1, 2, 5, 6$. In

Section 5 we find an exceptional manifold associated to the region X_4 and show that its fundamental group is isomorphic to the marked group G_4. This manifold, which we denote by N_4, is commensurable to the SnapPea census manifold $m369(-1,3)$. Lipyanskiy has described a sixth exceptional manifold in [13].

In Section 6, using Groebner bases we show that the quasi-relators have a unique solution in every region. Let N_0 =Vol3, $N_1 = v2678(2,1)$, $N_2 = s778(-3,1)$, N_3 be the exceptional manifold associated to X_3 found in [13], N_4 be exceptional manifold associated to X_4 found in Section 5 and $N_5 = s479(-3,1)$. We shall prove the following theorem.

Theorem 1. *Let N be an exceptional manifold. Then N is covered by N_i for some $i = 0,1,2,3,4,5$.*

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2. Invariant trace fields and 2-generator subgroups

Two finite volume, orientable, hyperbolic 3-manifolds are said to be *commensurable* if they have a common finite-sheeted cover. Subgroups $G, G' \subset \mathrm{PSL}(2,\mathbb{C})$ are *commensurable* if there exists $g \in \mathrm{PSL}(2,\mathbb{C})$ such that $g^{-1}Gg \cap G'$ is a finite index subgroup of both $g^{-1}Gg$ and G'. It follows by Mostow rigidity that finite volume, orientable, hyperbolic 3-manifolds are commensurable if and only if their fundamental groups are commensurable as subgroups of $\mathrm{PSL}(2,\mathbb{C})$. Let G be a group of covering transformations and let $\tilde{G}$ be its preimage in $\mathrm{SL}(2,\mathbb{C})$. It is shown in [14] that the traces of elements of $\tilde{G}$ generate a number field $\mathbb{Q}(\mathrm{tr}G)$ called the *trace field* of G. The *invariant trace field* $k(G)$ of G is defined as the intersection of all the fields $\mathbb{Q}(\mathrm{tr}H)$ where H ranges over all finite index subgroups of G. The definition already makes clear that the $k(G)$ is a commensurability invariant. In [17] Alan Reid proved the following result.

Theorem 2. *The invariant trace field $k(G)$ is equal to*

$$\mathbb{Q}(\{\mathrm{tr}^2(g) : g \in G\}) = \mathbb{Q}(\mathrm{tr}G^{(2)}),$$

where $G^{(2)}$ is the finite index subgroup of G generated by squares $\{g^2 : g \in G\}$.

From Corollary 3.2 of [10] the invariant trace field of a 2-generator group $\langle f, w | r_1, r_2\rangle$ is generated by $\mathrm{tr}(f^2), \mathrm{tr}(w^2)$ and $\mathrm{tr}(f^2w^2)$.

Using trace relations (see Theorem 4.2 of [4]) and Corollary 3.2 of [10], which says that $\mathbb{Q}(\mathrm{tr}G^{(2)}) = \mathbb{Q}(\mathrm{tr}G^{SQ})$ where $G^{SQ} = \langle g_1^2, \dots g_n^2\rangle$ with g_i's

generators of G such that $\mathrm{tr}(g_i) \neq 0$, we see that the invariant trace field of a 2-generator group $\langle f, w | r_1, r_2\rangle$ is generated by $\mathrm{tr}(f^2), \mathrm{tr}(w^2)$ and $\mathrm{tr}(f^2w^2)$.

As described in the Introduction, a marked group G is generated by f and w, where f and w are seen as covering transformations such that f represents the shortest geodesic and w takes the axis of f to its nearest translate. Let $\mathbb{H}^3$ denote the upper half space model of hyperbolic 3-space and the sphere at infinity be the $x-y$ plane. Conjugate G so that the axis of f is the geodesic line $B_{(0,\infty)}$ in $\mathbb{H}^3$ with end points 0 and ∞ on the sphere at infinity and the geodesic line perpendicular to $w^{-1}(B_{(0,\infty)})$ and $B_{(0,\infty)}$ (orthocurve) lies on the geodesic line $B_{(-1,1)}$ in $\mathbb{H}^3$ with endpoints -1 and 1 on the sphere at infinity. We can parametrize such a marked group with 3 complex numbers L, D and R where f is an L translation of $B_{(0,\infty)}$ and w is a D translation of $B_{(-1,1)}$ followed by an R translation of $B_{(0,\infty)}$. We can write matrix representatives for f and w using the exponentials L', D' and R' of L, D and R respectively (see Chapter 1 of [6]). We have

$$f = \begin{pmatrix} \sqrt{L'} & 0 \\ 0 & 1/\sqrt{L'} \end{pmatrix}, \qquad w = \begin{pmatrix} \sqrt{R'} * ch & \sqrt{R'} * sh \\ sh/\sqrt{R'} & ch/\sqrt{R'} \end{pmatrix} \tag{1}$$

where $ch = (\sqrt{D'} + 1/\sqrt{D'})/2$ and $sh = (\sqrt{D'} - 1/\sqrt{D'})/2$ We can write down the generators of the invariant trace field of G in terms of L', D' and R' as follows

$$\begin{aligned} \mathrm{tr}(f^2) &= L' + \frac{1}{L'} &(2)\\ \mathrm{tr}(w^2) &= \frac{1}{4}\Big[\Big(R' + \frac{1}{R'} + 2\Big)\Big(D' + \frac{1}{D'} + 2\Big) - 8\Big] &(3)\\ \mathrm{tr}(f^2w^2) &= \frac{1}{4}\Big[\Big(D' + \frac{1}{D'} + 2\Big)\Big(R'L' + \frac{1}{R'L'}\Big) \\ &\quad + \Big(D' + \frac{1}{D'} - 2\Big)\Big(L' + \frac{1}{L'}\Big)\Big] &(4) \end{aligned}$$

3. Guessing the algebraic numbers and exact arithmetic

In this section we find the marked groups for the regions as subgroups of $\mathrm{PSL}(2,\mathbb{C})$ with algebraic entries and find their invariant trace fields. In [6] parameter ranges for the seven regions are specified. For example, for the parameter range for region X_3 see Table 1.

We solve for L', D' and R' such that the quasi-relators are actually relations in the group. We obtain eight equations in three complex variables out of which three are independent and we use Newton's method with the parameter range as approximate solutions to find high precision solutions, e.g 100 significant digits, for the parameters satisfying the equations. This allows us to compute $a = \sqrt{L'}$, $b = \sqrt{R'}$, $c = \sqrt{D'}$ and $\mathrm{tr}(f^2)$, $\mathrm{tr}(w^2)$, $\mathrm{tr}(f^2w^2)$ to

high precision. Once the numbers are obtained to high precision we use the algdep() function of the PARI-GP package [8] to guess a polynomial over the integers that has the desired number as a root. Although the algdep() function cannot prove that the guess is in fact correct, we prove this by using the guessed values to perform exact arithmetic and verify the relations.

Once we obtain $a = \sqrt{L'}$, $b = \sqrt{R'}$, and $c = \sqrt{D'}$ as roots of polynomials we find a primitive element which generates the field that contains all the three numbers.

For the regions X_0, X_5, and X_6, a, b, and c are all contained in $\mathbb{Q}(a)$. By expressing the matrix entries as algebraic numbers one can verify the relations directly. For example, for X_0, the minimal polynomial for a and c is $x^8 + 2x^6 + 6x^4 + 2x^2 + 1$, and $b = 1$, so we can express a, b, and c as follows:

```
a = Mod(x, x^8 + 2*x^6 + 6*x^4 + 2*x^2 + 1),
b = Mod(1, x^8 + 2*x^6 + 6*x^4 + 2*x^2 + 1),
c = Mod(x, x^8 + 2*x^6 + 6*x^4 + 2*x^2 + 1).
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Then, using the formulae of Section 2, PARI-GP calculates the quasirelators exactly as follows:

```
[Mod(1, x^8 + 2*x^6 + 6*x^4 + 2*x^2 + 1) 0]
[0 Mod(1, x^8 + 2*x^6 + 6*x^4 + 2*x^2 + 1)].
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Thus, exact arithmetic verifies rigorously that the L', D', and R' which were calculated for X_0 using Newton's method are correct and the quasi-relators are in fact relators.

In general, the group coefficient field can have arbitrary index over the trace field. In order to keep the degree of the group coefficient field low we follow the method described in [13]. Given that f, w are generic ($fw - wf$ is nonsingular), if f_2, w_2 are any matrices in $\mathrm{SL}(2,\mathbb{C})$ such that $\mathrm{tr}(f_2) = \mathrm{tr}(f)$, $\mathrm{tr}(w_2) = \mathrm{tr}(w)$ and $\mathrm{tr}(f_2^{-1}w_2) = \mathrm{tr}(f^{-1}w)$ then the two pairs are conjugate.
Let $\mathrm{tr}_1 = \mathrm{tr}(f)$, $\mathrm{tr}_2 = \mathrm{tr}(w)$, $\mathrm{tr}_3 = \mathrm{tr}(f^{-1}w)$. Furthermore let

$$f_2 = \begin{pmatrix} 0 & 1 \\ -1 & \mathrm{tr}_1 \end{pmatrix}, \qquad w_2 = \begin{pmatrix} z & 0 \\ \mathrm{tr}_1 * z - \mathrm{tr}_3 & \mathrm{tr}_2 - z \end{pmatrix}, \tag{5}$$

where $(\mathrm{tr}_2 - z) * z = 1$. Then the pair (f_2, w_2) is conjugate to (f, w). The coefficients of the original f and w may have arbitrary index over the trace field but in this form the entries of the matrices are in an at most degree two extension of the trace field. Table 3 and 4 displays the computation of z, tr_1, tr_2 and tr_3 for all regions. In all cases z is the primitive element and

Region	Minimal Polynomial	Numerical Value
X_0	$\tau^8+2\tau^6+6\tau^4+2\tau^2+1$	$0.853230697 - 1.252448658i$
X_1	$\tau^8-2\tau^7+5\tau^6-4\tau^5+7\tau^4-4\tau^3+5\tau^2-2\tau+1$	$0.904047196 - 1.471654224i$
X_2	$\tau^4-2\tau^3+4\tau^2-2\tau+1$	$0.742934136 - 1.529085514i$
X_3	$\tau^{24} - 8\tau^{23} + 35\tau^{22} - 107\tau^{21} + 261\tau^{20} - 538\tau^{19}+972\tau^{18}-1565\tau^{17}+2282\tau^{16}-3034\tau^{15}+3706\tau^{14} - 4171\tau^{13} + 4339\tau^{12} - 4171\tau^{11} + 3706\tau^{10}-3034\tau^9+2282\tau^8-1565\tau^7+972\tau^6-538\tau^5+261\tau^4-107\tau^3+35\tau^2-8\tau+1$	$1.404292212 - 1.179267298i$
X_4	$\tau^6-3\tau^5+5\tau^4-4\tau^3+5\tau^2-3\tau+1$	$1.354619901 - 1.225125454i$
X_5	$\tau^{12}+2\tau^{10}+7\tau^8-4\tau^6+7\tau^4+2\tau^2+1$	$0.868063287 - 1.460023666i$
X_6	$\tau^{12}-2\tau^{10}+7\tau^8+4\tau^6+7\tau^4-2\tau^2+1$	$1.460023666 - 0.868063287i$

TABLE 3. Field containing z for all regions

$\mathrm{tr}_i \in \mathbb{Q}(z)$. One easily verifies the relations using the tables. We have the following theorem.

Theorem 3. *The marked groups G_i are 2-generator subgroups of* $\mathrm{PSL}(2,\mathbb{C})$ *with entries in the number-fields as given in Tables 3 and 4. Furthermore, the quasi-relators are relations in these groups.*

Remark 1. *It is proved in* [13] *that the quasi-relators generate all the relations for these groups and that the groups G_i are discrete co-compact subgroups of* $\mathrm{PSL}(2,\mathbb{C})$.

Remark 2. *f_2 and w_2 give an efficient way to solve the word problem in these groups.*

In this way we also obtain $\mathrm{tr}(f^2), \mathrm{tr}(w^2)$ and $\mathrm{tr}(f^2w^2)$ as roots of polynomials and find a primitive element which generates the field that contains all the three traces. We have the following theorem.

Theorem 4. *The invariant trace fields for all the regions are as given in Table 5.*

Remark 3. *The invariant trace field descriptions in Table 5 for X_i for $i \neq 3$ are the canonical field description given by Snap (see* [4]*).*

Region	tr_1	tr_2	tr_3
X_0	$-z-6z^3-2z^5-z^7$	tr_1	$(-5z^2-2z^4-z^6)/2$
X_1	$2-4z+4z^2-7z^3+4z^4-5z^5+2z^6-z^7$	tr_1	tr_1
X_2	$2-3z+2z^2-z^3$	tr_1	tr_1
X_3	$8-34z+107z^2-261z^3+538z^4-972z^5+1565z^6-2282z^7+3034z^8-3706z^9+4171z^{10}-4339z^{11}+4171z^{12}-3706z^{13}+3034z^{14}-2282z^{15}+1565z^{16}-972z^{17}+538z^{18}-261z^{19}+107z^{20}-35z^{21}+8z^{22}-z^{23}$	tr_1	tr_1
X_4	$3-4z+4z^2-5z^3+3z^4-z^5$	tr_1	tr_1
X_5	$-z-7z^3+4z^5-7z^7-2z^9-z^{11}$	tr_1	$(-6z^2+4z^4-7z^6-2z^8-z^{10})/2$
X_6	$3z-7z^3-4z^5-7z^7+2z^9-z^{11}$	tr_1	$(4-6z^2-4z^4-7z^6+2z^8-z^{10})/2$

TABLE 4. Group coefficients as polynomials in z in respective field

Region	Minimal Polynomial	Numerical Value
X_0	τ^2+3	$1.732050808i$
X_1	$\tau^4-2\tau^3+\tau^2-2\tau+1$	$-0.207106781+0.978318343i$
X_2	τ^2+1	i
X_3	$\tau^{12}+6\tau^{11}+23\tau^{10}+91\tau^9+257\tau^8+489\tau^7+823\tau^6+1054\tau^5-13\tau^4-2445\tau^3-3405\tau^2-1847\tau-337$	$0.632778000-3.019170376i$
X_4	$\tau^3-\tau-2$	$-0.760689853+0.857873626i$
X_5	$\tau^3-\tau^2+\tau+1$	$0.771844506+1.11514250i$
X_6	$\tau^3-\tau^2+\tau+1$	$0.771844506-1.11514250i$

TABLE 5. Invariant trace fields for all the regions

4. The manifolds for the regions X_1, X_2, X_5 and X_6

In this section we find manifolds from the Hodgson & Weeks census of closed hyperbolic 3-manifolds whose fundamental groups are isomorphic to

Region	V	H_1	l_{min}	Manifolds
X_1	4.11696874	$\mathbb{Z}_7 \oplus \mathbb{Z}_7$	1.0930	$v2678(2,1), v2796(1,2)$
X_2	3.66386238	$\mathbb{Z}_4 \oplus \mathbb{Z}_{12}$	1.061	$s778(-3,1), v2018(2,1)$
X_4	7.517689	$\mathbb{Z}_4 \oplus \mathbb{Z}_{12}$	1.2046	NA
X_5 or X_6	3.17729328	$\mathbb{Z}_4 \oplus \mathbb{Z}_4$	1.0595	$s479(-3,1), s480(-3,1),$ $s645(1,2), s781(-1,2),$ $v2018(-2,1)$

TABLE 6. Data for regions X_1, X_2, X_4, X_5 and X_6

Manifold	π_1 Relators
$v2678(2,1)$	$q_1 = a^2b^2aba^{-1}ba^{-1}b^{-1}a^{-1}ba^{-1}bab^2$ $q_2 = ab^{-1}ab^{-1}a^{-1}b^{-1}ab^{-1}aba^2b^2a^2b$
$s778(-3,1)$	$q_1 = ab^{-1}aba^2b^2ab^2a^2bab^{-1}$ $q_2 = ab^2a^2ba^2b^2aba^{-1}ba^{-1}b$
$s479(-3,1)$	$q_1 = aba^2b^2a^2bab^{-2}a^{-2}b^{-2}$ $q_2 = a^2b^2ab^2a^2bab^{-1}ab^{-1}ab$

TABLE 7. Relators for manifolds

the groups G_i for $i = 1, 2, 5, 6$. This census is included in Jeff Weeks' program SnapPea [18] and is refered to as SnapPea's census of closed hyperbolic 3-manifolds. These manifolds are described as Dehn surgeries on cusped hyperbolic 3-manifolds from SnapPea's census of cusped manifolds [1], [9]. We use the invariant trace fields and volume estimates for the regions given in [12] and [13] to search through the roughly 11,000 manifolds in the closed census. The package Snap [7] includes a text file called closed.fields which lists the invariant trace fields for the manifolds in the closed census. Using this file to compare the invariant trace fields we narrowed our search to less than 50 manifolds for each regions. Then using the homology, volume estimates and length of shortest geodesic we further narrowed the search to less than 5 manifolds. Table 6 gives the approximate volume (V) as given in [12], first homology (H_1), approximate length of shortest geodesic which is the values of the parameter L (l_{min}) and the manifold description as given in SnapPea.

The above manifolds include the manifolds mentioned in [6] for the regions X_1, X_2 and X_5. All the SnapPea manifolds associated to a region in Table 6 are isometric. It is shown in [12] that the manifolds associated to the regions X_5 and X_6 are isometric with an orientation reversing isomtery. The manifold associated to X_4 is discussed in the next section and that for X_3 is discussed in [13].

Region	Manifolds	Isomorphism	Inverse
X_1	$v2678(2,1)$	$f \longrightarrow a^{-1}$, $w \longrightarrow b$	$a \longrightarrow f^{-1}$, $b \longrightarrow w$
X_2	$s778(-3,1)$	$f \longrightarrow a$, $w \longrightarrow b^{-1}$	$a \longrightarrow f$, $b \longrightarrow w^{-1}$
X_5	$s479(-3,1)$	$f \longrightarrow ab$, $w \longrightarrow b$	$a \longrightarrow fw^{-1}$, $b \longrightarrow w$

TABLE 8. Isomorphisms

The fundamental groups of the above manifolds have two generators and two relators. Table 2 and 7 gives the relators for the marked groups and the relators for the fundamental groups of the corresponding manifolds respectively. One can verify the isomorphisms between the groups given in Table 8. We have the following theorem.

Theorem 5. *The manifolds $v2678(2,1)$, $s778(-3,1)$ and $s479(-3,1)$ in SnapPea's census of closed manifolds are exceptional manifolds associated to the regions X_1, X_2 and X_5 respectively.*

5. The manifold associated to the region X_4

In this section we give a description of the manifold associated to the region X_4 as a double cover of an orbifold commensurable to the manifold $m369(-1,3)$ in SnapPea's census of closed manifolds.

In Section 4 using the approximate volumes and other data given in [12] and [13] we found manifolds from the SnapPea's census of closed manifolds with fundamental groups isomorphic to the groups for the regions X_1, X_2, X_5 and X_6. The regions X_3 and X_4 could not be handled because of their large volumes. However for the region X_4 a list of manifolds was found in the closed census having approximately half the volume of X_4 and the same commensurability invariants.

In hope of obtaining the manifold for X_4 as a double cover of one of these manifolds we compared index two subgroups of the fundamental groups of each of these manifolds to G_4, the marked group for X_4. Most of the subgroups were eliminated on basis of homology, however one index two subgroup of the census manifold $m369(-1,3)$ had the correct homology, and same lengths for its elements as for X_4. Using the program *testisom* [11] it was checked that this subgroup was not isomorphic to G_4.

Theorem 6. *The manifold N_4 associated to the region X_4 is commensurable with the manifold $m369(-1,3)$ in SnapPea's census of closed manifolds. This manifold is obtained as a double cover of an orbifold which is double covered by a double cover of $m369(-1,3)$.*

Proof: Let $M = m369(-1,3)$. We will construct the following diagram of $2:1$ covers:

$$\begin{array}{ccc} N & & N_4 \\ \downarrow & \searrow & \downarrow \\ M & & O \end{array}$$

We obtain a presentation of $\pi_1(M)$ from SnapPea.

$$\pi_1(M) = \langle a,\ b,\ c |\ ab^{-1}a^{-1}c^2bc,\ abcb^3a^{-1}c^{-1},\ acbc^{-1}b^{-1}cbacb\rangle$$

Let $\phi : \pi_1(M) \to \mathbb{Z}_2$ be defined by $\phi(a) = 1, \phi(b) = \phi(c) = 0$. Then ϕ is a homomorphism and $\ker(\phi)$ is an index two subgroup of $\pi_1(M)$ generated by b and c. Let N denote the double cover of M corresponding to this subgroup so that $\pi_1(N) = \ker(\phi)$. A presentation of $\pi_1(N)$ is

$$\pi_1(N) = \langle b,\ c |\ r_1,\ r_2\rangle$$

where

$$\begin{aligned} r_1 &= bcb^3cbc^{-1}b^{-1}cbc^{-1}b^{-1}cbc^2(bc^3)^2bc^2bcb^{-1}c^{-1}bcb^{-1}c^{-1}, \\ r_2 &= cbc^{-1}b^{-1}cbc^{-1}b^{-1}cbc^2(bc^3)^2(bc^2bc^3bc^3bc^3)^2bcb^{-1}c^{-1}bcb^{-1}c^{-1}b. \end{aligned}$$

Let $\psi : \pi_1(N) \to \pi_1(N)$ be defined by $\psi(c) = c^{-1}$ and $\psi(b) = c^3b$. Then ψ is an automorphism of $\pi_1(N)$ of order two. Extending the group $\pi_1(N)$ by this automorphism we obtain a group H whose presentation is

$$H = \langle b,\ c,\ t |\ r_1,\ r_2,\ tct^{-1}c,\ tbt^{-1}b^{-1}c^{-3},\ t^2\rangle$$

$\pi_1(N)$ is a subgroup of H of index two and the quotient of $\mathbb{H}^3$ by H is an orbifold O (due to the torsion element t) which is double covered by N. Let $\mu : H \to \mathbb{Z}_2$ be defined by $\mu(c) = 0,\ \mu(b) = \mu(t) = 1$. Then μ is a homomorphism and $\ker(\mu)$ is an index 2 subgroup of H generated by elements c and $b * t$. Let $x = c$ and $y = b * t$. Then a presentation of $\ker(\mu) = G$ is

$$G = \ker(\mu) = \langle x,\ y |\ s_1,\ s_2,\ s_3\rangle$$

where

$$\begin{aligned} s_1 &= (yx^{-1}y^{-1}x^{-1})^2yx^2y^2x^3yxyxy^{-1}xyxy^{-1}xyxyx^3y^2x^2, \\ s_2 &= (yxy^{-1}x)^2yxyx^3(y^2x^2y^2x^3)^2yxyxy^{-1}xyxy^{-1}xyx, \\ s_3 &= y^{-1}x^{-3}(y^{-1}x^{-1})^2yx^{-1}y^{-1}x^{-1}yx^2y^2x^3yxyx^3y^2x^2(yx^{-1}y^{-1}x^{-1})^2. \end{aligned}$$

The presentation for the marked group G_4 as given in [6] is

$$G_4 = \langle f,\ w |\ r_1(X_4),\ r_2(X_4)\rangle$$

where

$$\begin{aligned} r_1(X_4) &= f^{-2}wfwf^{-1}(wfw^{-1}f)^2wf^{-1}wfwf^{-2}w^2(f^{-1}w^{-1}f^{-1}w)^2w, \\ r_2(X_4) &= f^{-1}(f^{-1}wfw)^2f^{-2}w^2f^{-1}w^{-1}f^{-1}w(f^{-1}w^{-1}fw^{-1})^2 \\ &\quad \times f^{-1}wf^{-1}w^{-1}f^{-1}w^2. \end{aligned}$$

One easily verifies that the map $\nu : G_4 \to G$ given by $\nu(x) = f$ and $\nu(y) = f^{-1}w^{-1}$ is an isomorphism. The inverse of ν is given by $\nu^{-1}(f) = y$ and $\nu^{-1}(w) = y^{-1}x^{-1}$. Lipyanskiy [13] constructed a Dirichlet domain for all the regions whose groups are isomorphic to the marked groups. It follows that G_4 is torsion free and hence G is a torsion free subgroup of H of index 2. Hence it gives the manifold N_4 which double covers the orbifold O.□

Remark 4. *The symmetries of the configuration of lines in $\mathbb{H}^3$ consisting of the axis of f, w, their translations and the orthocurves between them led us to study the above mentioned subgroups and automorphisms. N has a geodesic of the same length and the same translation length as N_4 but it is not the shortest geodesic in N.*

6. Uniqueness

In this section we address the issue of uniqueness of the solutions in the given region. In Chapter 3 of [6] the authors showed the existence and uniqueness of solution for the region X_0 by using a geometric argument to show $R' = 1$ and then using the symmetry of the region X_0 to reduce the number of variables and obtain a one variable equation which has only one solution in the region X_0. Using Groebner basis we show that there is a unique point in every region X_i for which the quasi-relators equal the identity.

Let I be the ideal generated by the equations formed by the entries of the quasi-relators of a region subtracted from the identity matrix. We compute a Groebner basis for I and verify that there is a unique solution to equations in the Groebner basis in that region. For computational convenience we split the relations in half as it reduces the degree of the polynomials generating the ideal. Let $p = \mathrm{tr}_1 = \mathrm{tr}(f)$, $q = \mathrm{tr}_2 = \mathrm{tr}(w)$ and $r = \mathrm{tr}_3 = \mathrm{tr}(f^{-1}w)$ as in Section 3. Using Equation 1 in Section 2, L', D' and R' can be expressed in terms of p, q and r as follows:

$$L' = \Big(\frac{p \pm \sqrt{p^2-4}}{2}\Big)^2 \tag{6}$$

$$D' = \Big(\frac{2q\sqrt{R'} \pm \sqrt{4q^2R' - 4(1+R')^2}}{2(1+R')}\Big)^2 \tag{7}$$

$$R' = \frac{qL' - r\sqrt{L'}}{r\sqrt{L'} - q} \tag{8}$$

Using Equation 5 in Section 3 we can write the entries of conjugates of f and w in terms of p, q, r and z where $qz - z^2 = 1$. The equations for quasi-relators using Equation 5 are simpler for computing Groebner basis.

For example, for the region X_0, using the ordering z, r, q, p on the variables, the last entry of the Groebner basis is $(p-1)(p+1)(p^4 - 2p^2 + 4)$. Using

Equation 6 it can be easily checked that only one root of the above equation gives the value of L' lying in the region X_0. Similarly the last entries of the Groebner bases in orders $z,\ r,\ p,\ q$ and $z,\ p,\ q,\ r$ are $(q-2)(q+2)(q^4-2q^2+4)$ and $(r+1)(r^2-r+1)$ respectively. Using Equations 7 and 8 it can be easily checked that only one root of q^4-2q^2+4 and r^2-r+1 give the value of D' and R' respectively lying in the region X_0. This shows that there is a unique solution for the quasi-relators in the region X_0.

Similarly for the regions $X_2,\ X_4,\ X_5$ and X_6 the last entry of the Groebner basis is a polynomial in either $p,\ q$ or r depending on the ordering of the variables. Using Equations 6, 7 and 8 we check that there is a unique solution in the respective region.

For the regions X_1 and X_3 we obtain a multivariable polynomial in p, q and r as a factor of the last entry of the Groebner basis along with a single variable polynomial. We eliminate this factor using the Mean Value Theorem.

For example, for the region X_3 the last entry of the Groebner basis with the ordering $z > r > q > p$ on the variables factors as:

$$(p^3+p^2-2p-1)(p^{10}-7p^9+15p^8+4p^7-49p^6+11p^5+88p^4+87p^3-501p^2+543p-193)(p^{10}+5p^9+6p^8-6p^7-10p^6+12p^5+13p^4-11p^3-6p^2+4p+1)(p^{12}-8p^{11}+23p^{10}-19p^9-35p^8+73p^7-3p^6-72p^5+25p^4+29p^3-11p^2-3p+1)(rq+rp-r+qp-q-p+1)$$

It can be checked that only one root of the polynomial in p above gives the value of L' lying in the region X_3. We will show that the polynomial $rq+rp-r+qp-q-p+1$ has no root in the region X_3.

From Table 1, we see that

$$\begin{aligned} L' &= 0.581385 - 3.312055\, i, \\ D' &= 1.15663 - 2.756005\, i, \\ R' &= 1.40437 - 1.179435\, i \end{aligned}$$

is the midpoint of region X_3. Then using Equation 1 the $p,\ q,\ r$ values corresponding to this point are

$$\begin{aligned} p_0 &= 1.8219 - 0.828571\, i, \\ q_0 &= 1.82191 - 0.828633\, i, \\ r_0 &= 1.82192 - 0.828537\, i \end{aligned}$$

If the polynomial $f(p,q,r) = rq+rp-r+qp-q-p+1$ has a root, say (p_1,q_1,r_1), in the region X_3 then by the Mean Value Theorem for some point (p,q,r) in X_3 we obtain

$$\begin{aligned} |f(p_0,q_0,r_0)| &= |\nabla f(p,q,r)\cdot(p_1-p_0, q_1-q_0, r_1-r_0)| \\ &\leq ||\nabla f(p,q,r)||\,||(p_1-p_0, q_1-q_0, r_1-r_0)|| \end{aligned}$$

From the parameter ranges of region X_3 from Table 1 we know that $||(p_1 - p_0, q_1 - q_0, r_1 - r_0)|| < 0.002$. Hence $||\nabla f(p,q,r)|| \geq |f(p_0,q_0,r_0)|/0.002$ at some point in the region X_3. It can be checked that $|f(p_0,q_0,r_0)|/0.002 \sim 3000$ and that

$$\begin{aligned} ||\nabla(f)|| &\leq \sqrt{(|p|+|q|+1)^2+(|q|+|r|+1)^2+(|r|+|p|+1)^2} \\ &\leq \sqrt{(|p_0|+|q_0|+2)^2+(|q_0|+|r_0|+2)^2+(|r_0|+|p_0|+2)^2} \\ &< 11 \end{aligned}$$

in the region X_3 and hence $f(p,q,r)$ does not have a root in X_3.

We can similarly check for the other variables by changing the order of the variables. The last entry of the Groebner basis with the ordering $z > r > p > q$ on the variables factors as

$(q-2)(q^{10}-7q^9+15q^8+4q^7-49q^6+11q^5+88q^4+87q^3-501q^2+543q-193)(q^{10}+5q^9+6q^8-6q^7-10q^6+12q^5+13q^4-11q^3-6q^2+4q+1)(q^{12}-8q^{11}+23q^{10}-19q^9-35q^8+73q^7-3q^6-72q^5+25q^4+29q^3-11q^2-3q+1)(rp+rq-r+pq-p-q+1)$

and the last entry of the Groebner basis with the ordering $z > p > q > r$ on the variables factors as

$(r^3+r^2-2r-1)(r^{10}-7r^9+15r^8+4r^7-49r^6+11r^5+88r^4+87r^3-501r^2+543r-193)(r^{10}+5r^9+6r^8-6r^7-10r^6+12r^5+13r^4-11r^3-6r^2+4r+1)(r^{12}-8r^{11}+23r^{10}-19r^9-35r^8+73r^7-3r^6-72r^5+25r^4+29r^3-11r^2-3r+1)(pq+pr-p+qr-q-r+1)$

Using Equations 6, 7 and 8 above it can be checked that there is only one root of the above polynomials in q and r which give values for D' and R' respectively lying in the region X_3. The polynomial in p, q and r is the same as above. Hence the quasi-relators for the region X_3 have a unique solution in the region X_3. The uniqueness of solutions is proved similarly for the region X_1.

Proposition 1. *Let f and w be as in Equation 1 and let $r_1(X_i)$, $r_2(X_i)$ be quasi-relators for the region X_i. Then there is a unique triple (L', D', R') in the region X_i for which the quasi-relators equal the identity matrix.*

Proof: It follows from Theorem 3 that there is a triple (L', D', R') in the region X_i for which the quasi-relators equal the identity matrix. The uniqueness follows from the Groebner basis computation for every region. □

We now give the proof of our main Theorem.

Proof of Theorem 1: Let N be an exceptional manifold and let δ be

the shortest geodesic in N. Let $f \in \pi_1(N)$ be a primitive hyperbolic isometry whose fixed axis $\delta_0 \in \mathbb{H}^3$ projects to δ and $w \in \pi_1(N)$ be a hyperbolic isometry which takes δ_0 to its nearest translate. Let G be the subgroup of $\pi_1(N)$ generated by f and w. Then the manifold $N' = \mathbb{H}^3/G$ is exceptional and δ_0 projects to the shortest geodesic in N'.

It follows from Corollary 1.29 of [6] that the (L', D', R') parameter for G lies in the region X_i for some $i = 0, 1, \ldots, 6$. By definition of the quasi-relators [6], $Relength(r_1)$, $Relength(r_2) < Relength(f)$. Since f is the shortest element in G, r_1 and r_2 equal the identity in G i.e. they are relations in G. It is proved in [13] that the quasi-relators generate all the relations in the groups $G_i = \langle f, w | r_1(X_i), r_2(X_i)\rangle$ for $i = 0, \ldots, 6$. Hence $G = G_i$ and $\pi_1(N)$ contains the marked group G_i for some $i = 0, 1, \ldots 6$.

It follows from Proposition 1 that the quasi-relators for a given region equal the identity at a unique point in that region. Hence N is covered by N_i for some $i = 0, 1, 2, 3, 4, 5$ where N_i are the manifolds described in the Introduction. □

7. Conclusions

The first part of Conjecture 1 follows from Proposition 1. The results in Section 4 and 5 shows part (i) of Conjecture 1 for regions X_i, $i = 1, 2, 4, 5, 6$ and the exact arithmetic from Section 3 show part (v) of Conjecture 1 for all the regions. The question about the uniqueness of the manifolds remains open for all regions except X_0. It is reasonable to make the following conjecture:

Conjecture 2. *The manifolds N_i for $i = 1, 2, 3, 4, 5, 6$ do not nontrivially cover any manifold.*

Alan Reid proves the conjecture for N_1 and N_5 in the following appendix.

8. Appendix: The manifolds N_1 and N_5

Alan W. Reid

In this appendix we prove the following theorem.

Theorem. *The manifolds N_1 and N_5 do not properly cover any closed hyperbolic 3-manifold*

Proof: We give the proof in the case of N_1, the case of N_5 is similar. Both arguments follow that given for Vol3 in [12]. We refer the reader to [16] for details about arithmetic Kleinian groups and quaternion algebras.

Thus, suppose that $N_1 = \mathbb{H}^3/\Gamma_1$ non-trivially covered a closed hyperbolic 3-manifold $N = \mathbb{H}^3/\Gamma$ say, with covering degree d. Using the identification of N_1 as $v2678(2,1)$ given in the paper, it follows that the volume of N_1 is approximately $4.116968736384613\ldots$ and $H_1(N_1;\mathbb{Z}) = \mathbb{Z}/7\mathbb{Z} \oplus \mathbb{Z}/7\mathbb{Z}$ is finite of odd order. Note that since $H_1(N_1;\mathbb{Z})$ is finite the closed hyperbolic 3-manifold N is orientable. Since the volume of the smallest arithmetic manifold is approximately 0.94 [3], it follows that $d \leq 4$.

Using Snap (or from computations in the paper) the Kleinian group Γ_1, and hence Γ, is arithmetic with invariant trace-field k say. This has degree 4, has discriminant -448, and the invariant quaternion algebra B/k unramified at all finite places. We remark that there is a unique such field.

Since $|H_1(N_1;\mathbb{Z})|$ is odd, Γ_1 is derived from a quaternion algebra. Furthermore, B has type number 1, and so Γ_1 is conjugate into the group of elements of norm 1 of a maximal order $\mathcal{O}$ of B. Now the image of the elements of norm 1 of $\mathcal{O}$ in $\mathrm{PSL}(2,\mathbb{C})$ can be shown (see [15]) to coincide with the orientation preserving subgroup of index 2 in the Coxeter simplex group $T[2,3,3;2,3,4]$. The notation for the Coxeter group is that of [15].

Denote this group by C. The minimal index of a torsion free subgroup of C is at least 24, since by inspection of the Coxeter diagram, C contains a subgroup isomorphic to S_4. Therefore the volume calculations of [15] show that Γ_1 is a minimal index torsion-free subgroup of this group.

Now the analysis in §4 of [12] shows that the possible maximal groups in the commensurabilty class of Γ_1 that contain Γ are either the group $\Gamma_{\mathcal{O}}$ (in the notation [12]) where $\mathcal{O}$ is the maximal order above, or $\Gamma_{\{\mathcal{P}_7\},\mathcal{O}}$ (in the notation of [12]).

Suppose first that $\Gamma < \Gamma_{\mathcal{O}}$. By the remarks above Γ is not a subgroup of C. Now [15] shows that $\Gamma_{\mathcal{O}}$ contains C as a subgroup of index 2. It follows that Γ must contain Γ_1 as a subgroup of index 2, and that Γ is a torsion-free subgroup of index 24 in $\Gamma_{\mathcal{O}}$. However we claim that this is impossible.

Firstly we can obtain a presentation of the group $\Gamma_{\mathcal{O}}$ using the geometric description of C above; namely the group $\Gamma_{\mathcal{O}}$ is obtained from C by adjoining an orientation-preserving involution t that is visible in the Coxeter diagram. On checking the action of this involution, one gets that a presentation for $\Gamma_{\mathcal{O}}$ is given by:

$< t,a,b,c \mid t^2,\ a^2,\ b^3,\ c^3,\ (b*c)^2,\ (c*a)^3,\ (a*b)^4,\ t*a*t^{-1}*c*b,\ t*b*t^{-1}*a*c^{-1},\ t*c*t^{-1}*c >$.

Now a check with Magma (for example) shows that there are 24 subgroups of index 24 and all but two are easily seen to have elements of finite order by inspection of presentations. The remaining two have abelianizations $\mathbb{Z}/22\mathbb{Z}$.

The index 2 subgroups in these groups all have $\mathbb{Z}/11\mathbb{Z}$ in their abelianizations by a standard cohomology of groups argument (or further checking with Magma). In particular these index 2 subgroups cannot coincide with Γ_1, which completes the analysis in this case.

For the second case the covolume of $G = \Gamma_{\{\mathcal{P}_7\},\mathcal{O}}$ can be computed (using the formula in §2 of [12]) to be 12 times that of Γ_1. An alternative, equivalent description of this maximal group, is as the normalizer of an Eichler order $\mathcal{E}$ of level $\mathcal{P}_7$ in $\mathcal{O}$ (see [16] Chapter 11 for example).

The results of [2] (see in particular Theorems 3.3 and 3.6) show that G contains elements of orders 2 and 3, and so the minimal index of a torsion-free subgroup in G is at least 6.

Now $\Gamma_1 \subset G \cap C$. Furthermore, if we denote the image of the group of elements of norm 1 in $\mathcal{E}$ in $\mathrm{PSL}(2,\mathbb{C})$ by $\Gamma^1_{\mathcal{E}}$, then since the level is $\mathcal{P}_7$ it follows that the index $[C : \Gamma^1_{\mathcal{E}}]$ is 8 (see [16] Chapters 6 and 11). It is not difficult to see that $G \cap C = \Gamma^1_{\mathcal{E}}$. One inclusion is clear, and the other follows since, from above $[C : \Gamma^1_{\mathcal{E}}] = 8$ so that the only possible indices for $[C : G \cap C]$ are 2 or 4 (it cannot be 1 since G is a different maximal group from $\Gamma_{\mathcal{O}}$ above). Now C is perfect, and so has no solvable quotients. Hence this rules out C from having index 2 or 4 subgroups.

We deduce from the above that Γ_1 is a subgroup of $\Gamma^1_{\mathcal{E}}$. Using the presentation of C, and on checking with Magma for instance, we find that there are 5 subgroups of index 8, and some further low index computations on these subgroups using Magma shows that only one can contain Γ_1. This subgroup (denoted H in what follows) is generated by two elements of order 3 (b and ca in the generators above).

As in the first case we can use the geometry associated to H to construct a presentation for the group G. H is 2-generator with both generators of order 3, so we can adjoin involutions s and t so that a presentation for G is:

$$< x,y,s,t \mid s^2,\ t^2, (s*t)^2,\ s*a*s*y^{-1}, s*y*s^{-1}*x^{-1}, t*x*t*x,\ t*y*t^{-1}*y,\ x^3,\ y^3,\ x*y^{-1}*x^{-1}*y*x*y*x^{-1}*y^{-1}*x*y*x*y^{-1}*x^{-1}*y^{-1}*x*y*x^{-1}*y^{-1}*x^{-1}*y*x*y^{-1}*x^{-1}*y^{-1}*x*y > .$$

From our remarks above we need only check for torsion-free subgroups of index 6. However, an easy inspection using Magma shows that all the subgroups of index 6 (there are 4) have elements of finite order. Hence this completes the proof. □

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References

[1] Patrick J. Callahan, Martin V. Hildebrand, and Jeffrey R. Weeks, *A census of cusped hyperbolic* 3*-manifolds*, Math. Comp. **68** (1999), no. 225, 321–332, With microfiche supplement. MR 99c:57035

[2] T. Chinburg and E. Friedman, *The finite subgroups of maximal arithmetic Kleinian groups*, Ann. Inst. Fourier (Grenoble) **50** (2000), no. 6, 1765–1798 (2001). MR MR1817383 (2002g:11162)

[3] Ted Chinburg, Eduardo Friedman, Kerry N. Jones, and Alan W. Reid, *The arithmetic hyperbolic 3-manifold of smallest volume*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) **30** (2001), no. 1, 1–40. MR MR1882023 (2003a:57027)

[4] David Coulson, Oliver A. Goodman, Craig D. Hodgson, and Walter D. Neumann, *Computing arithmetic invariants of 3-manifolds*, Experiment. Math. **9** (2000), no. 1, 127–152. MR 2001c:57014

[5] David Gabai, *On the geometric and topological rigidity of hyperbolic* 3*-manifolds*, J. Amer. Math. Soc. **10** (1997), no. 1, 37–74. MR 97h:57028

[6] David Gabai, G. Robert Meyerhoff, and Nathaniel Thurston, *Homotopy hyperbolic 3-manifolds are hyperbolic*, Ann. of Math. (2) **157** (2003), no. 2, 335–431. MR 2004d:57020

[7] Oliver A. Goodman, Craig D. Hodgson, and Walter D. Neumann, *Snap Home Page*, Available online from http://www.ms.unimelb.edu.au/∼snap/ (1998).

[8] The PARI Group, *Pari/GP, version* `2.1.3`, Available online from http://pari.math.u-bordeaux.fr/ (2002).

[9] Martin Hildebrand and Jeffrey Weeks, *A computer generated census of cusped hyperbolic* 3*-manifolds*, Computers and mathematics (Cambridge, MA, 1989), Springer, New York, 1989, pp. 53–59. MR 90f:57043

[10] Hugh M. Hilden, María Teresa Lozano, and José María Montesinos-Amilibia, *A characterization of arithmetic subgroups of* $\mathrm{SL}(2,\mathbf{R})$ *and* $\mathrm{SL}(2,\mathbf{C})$, Math. Nachr. **159** (1992), 245–270. MR 94i:20088

[11] Derek Holt and Sarah Rees, *isom_quotpic*, Available online from ftp://ftp.maths.warwick.ac.uk/people/dfh/isom_quotpic/ (1997).

[12] Kerry. N. Jones and Alan. W. Reid, *Vol3 and other exceptional hyperbolic 3-manifolds*, Proc. Amer. Math. Soc. **129** (2001), no. 7, 2175–2185 (electronic). MR 2002e:57024

[13] Max Lipyanskiy, *A computer-assisted application of Poincare's fundamental polyhedron theorem*, Preprint (2002).

[14] A. Murray Macbeath, *Commensurability of co-compact three-dimensional hyperbolic groups*, Duke Math. J. **50** (1983), no. 4, 1245–1253. MR 85f:22013

[15] C. Maclachlan and A. W. Reid, *The arithmetic structure of tetrahedral groups of hyperbolic isometries*, Mathematika **36** (1989), no. 2, 221–240 (1990). MR MR1045784 (91b:11055)

[16] Colin Maclachlan and Alan W. Reid, *The arithmetic of hyperbolic 3-manifolds*, Graduate Texts in Mathematics, vol. 219, Springer-Verlag, New York, 2003. MR MR1937957 (2004i:57021)

[17] Alan W. Reid, *A note on trace-fields of Kleinian groups*, Bull. London Math. Soc. **22** (1990), no. 4, 349–352. MR 91d:20056

[18] Jeffery R. Weeks, *SnapPea*, Available online from http://www.northnet.org/weeks (1993).

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